

A Second Study of the Effects of a Non-Sinusoidal Motor

J. Borders
5-21-2008

Throughout this analysis, please refer to Figure 1 for voltage polarities, sign conventions, etc. The self-inductances of the windings are assumed to be negligible for this analysis.

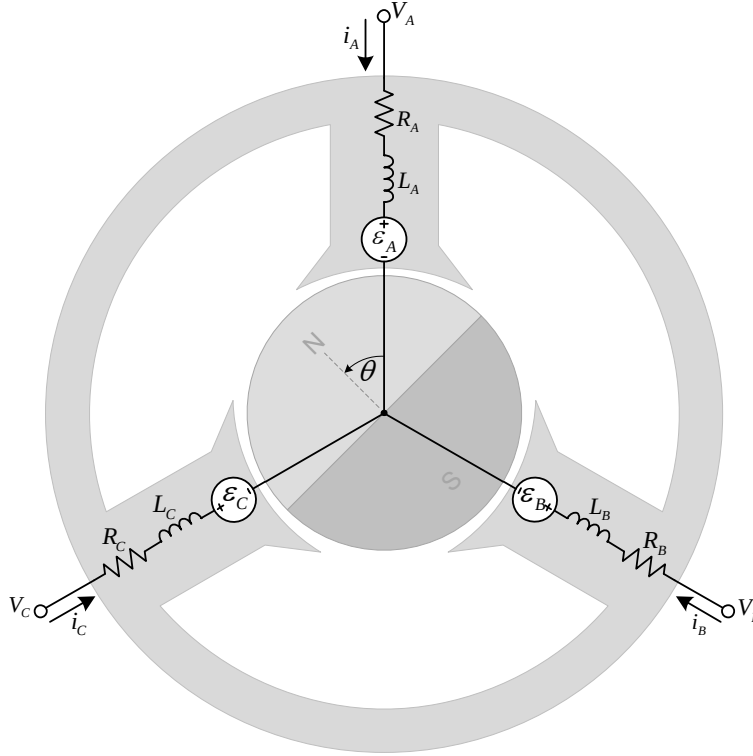


Figure 1. A diagram of a brushless DC motor

Formulation of non-sinusoidal winding EMF's and torques

Many times when analyzing brushless DC motors, we assume many things about the motor. For example, we'll often begin assuming the motor winding EMF's and torque profiles are given by:

$$\varepsilon_A = k_v \dot{\theta} \sin(\theta) \quad [1.1]$$

$$\varepsilon_B = k_v \dot{\theta} \sin(\theta + 120^\circ) \quad [1.2]$$

$$\varepsilon_C = k_v \dot{\theta} \sin(\theta + 240^\circ) \quad [1.3]$$

and:

$$T_A = k_t i_A \sin(\theta) \quad [1.4]$$

$$T_B = k_t i_B \sin(\theta + 120^\circ) \quad [1.5]$$

$$T_C = k_t i_C \sin(\theta + 240^\circ) \quad [1.6]$$

The above equations assume the motor is perfectly symmetric, and is "sinusoidal." But let's assume the motor winding EMF's and torque profiles are not sinusoidal. A much more general way of expressing them is as follows:

$$\varepsilon_A = k_v \dot{\theta} \cdot p(\theta) \quad [1.7]$$

$$\varepsilon_B = k_v \dot{\theta} \cdot p(\theta + 120^\circ) \quad [1.8]$$

$$\varepsilon_C = k_v \dot{\theta} \cdot p(\theta + 240^\circ) \quad [1.9]$$

and

$$T_A = k_t i_A \cdot p(\theta) \quad [1.10]$$

$$T_B = k_t i_B \cdot p(\theta + 120^\circ) \quad [1.11]$$

$$T_C = k_t i_C \cdot p(\theta + 240^\circ) \quad [1.12]$$

where $p(\theta)$ is a periodic function of period 2π and with zero average. Note that we are still enforcing a type of symmetry by assuming each of the three windings are 120 degrees apart. However, the shape of the curves is arbitrary, as long as they have a period of 2π .

We can express $p(\theta)$ in terms of its Fourier series:

$$p(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) \quad [1.13]$$

Note that in this form, each winding curve (EMF and torque) can be described in an infinite number of ways. To illustrate, we expand [1.7] using [1.13]:

$$\varepsilon_A = \dot{\theta} [k_v a_1 \sin(\theta) + k_v b_1 \cos(\theta) + k_v a_2 \sin(2\theta) + k_v b_2 \cos(2\theta) + \dots] \quad [1.14]$$

If every Fourier coefficient was increased (multiplied) by some constant factor, α , and k_v was decreased (divided) by the same factor, ε_A would remain the same. What we'd like is for k_v and k_t to define the "size" of the winding EMF and torque curves, and for the Fourier coefficients in [1.13] to define the "shape" of the curves. This requires us to in some way normalize $p(\theta)$. One (but by no means the only) way to do this is to impose the constraint:

$$a_1 = 1, \quad b_1 = 0 \quad [1.15]$$

Actually, this is two constraints, and it does more than normalize the "size" of $p(\theta)$ - it also normalizes its phase (which is fine for this analysis).

Determining Average Motor Torque from an (almost) Arbitrary Winding Current Waveform

Assume that a non-sinusoidal motor is being driven by a motor controller employing an unidentified commutation scheme. All that is available is experimental winding current data. This analysis will assume the winding currents are of the form:

$$i_A = q(\theta) \quad [1.16]$$

$$i_B = q(\theta + 120^\circ) \quad [1.17]$$

$$i_C = q(\theta + 240^\circ) \quad [1.18]$$

where,

$$q(\theta) = \sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \quad [1.19]$$

The implicit assumption built into these equations is that each of the three winding currents look exactly the same (in shape and amplitude), but are just shifted by 120 degrees from one another. In general, it is up to the motor controller to ensure this requirement is met. Also, note that in order to satisfy conservation of current, the following must hold:

$$i_A + i_B + i_C = 0 \Rightarrow q(\theta) + q(\theta + 120^\circ) + q(\theta + 240^\circ) = 0 \quad [1.20]$$

It's not obvious whether this is true for the most general form of $q(\theta)$. So let's take a look. Note the following:

$$\begin{aligned} q(\theta) + q(\theta + 120^\circ) &= \sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) + \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 120^\circ) + d_n \cos(n\theta + n \cdot 120^\circ) \\ &= \sum_{n=1}^{\infty} c_n [\sin(n\theta) + \sin(n\theta + n \cdot 120^\circ)] + d_n [\cos(n\theta) + \cos(n\theta + n \cdot 120^\circ)] \\ &= \sum_{n=1}^{\infty} \left\{ c_n [\sin(n\theta) + \sin(n\theta)\cos(n \cdot 120^\circ) + \cos(n\theta)\sin(n \cdot 120^\circ)] + \right. \\ &\quad \left. d_n [\cos(n\theta) + \cos(n\theta)\cos(n \cdot 120^\circ) - \sin(n\theta)\sin(n \cdot 120^\circ)] \right\} \quad [1.21] \\ &= \sum_{n=1}^{\infty} \left\{ c_n \sin(n\theta) + c_n \cos(n \cdot 120^\circ) \sin(n\theta) + c_n \sin(n \cdot 120^\circ) \cos(n\theta) + \right. \\ &\quad \left. d_n \cos(n\theta) + d_n \cos(n \cdot 120^\circ) \cos(n\theta) - d_n \sin(n \cdot 120^\circ) \sin(n\theta) \right\} \\ &= \sum_{n=1}^{\infty} [c_n + c_n \cos(n \cdot 120^\circ) - d_n \sin(n \cdot 120^\circ)] \sin(n\theta) + [c_n \sin(n \cdot 120^\circ) + d_n + d_n \cos(n \cdot 120^\circ)] \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \left\{ c_n [1 + \cos(n \cdot 120^\circ)] - d_n \sin(n \cdot 120^\circ) \right\} \sin(n\theta) + \left\{ c_n \sin(n \cdot 120^\circ) + d_n [1 + \cos(n \cdot 120^\circ)] \right\} \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \left\{ c_n [1 + \cos(n \cdot 240^\circ)] + d_n \sin(n \cdot 240^\circ) \right\} \sin(n\theta) + \left\{ -c_n \sin(n \cdot 240^\circ) + d_n [1 + \cos(n \cdot 240^\circ)] \right\} \cos(n\theta) \end{aligned}$$

Rearranging, we see that:

$$- [q(\theta) + q(\theta + 120^\circ)] = \sum_{n=1}^{\infty} \left\{ -c_n [1 + \cos(n \cdot 240^\circ)] - d_n \sin(n \cdot 240^\circ) \right\} \sin(n\theta) + \left\{ c_n \sin(n \cdot 240^\circ) - d_n [1 + \cos(n \cdot 240^\circ)] \right\} \cos(n\theta) \quad [1.22]$$

Also note that:

$$\begin{aligned} q(\theta + 240^\circ) &= \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 240^\circ) + d_n \cos(n\theta + n \cdot 240^\circ) \quad [1.23] \\ &= \sum_{n=1}^{\infty} [c_n \cos(n \cdot 240^\circ) - d_n \sin(n \cdot 240^\circ)] \sin(n\theta) + [c_n \sin(n \cdot 240^\circ) + d_n \cos(n \cdot 240^\circ)] \cos(n\theta) \end{aligned}$$

Seeing that [1.22] and [1.23] are both in the form of a Fourier series, we can compare their Fourier coefficients. And we see by inspection that in order for $-[q(\theta) + q(\theta + 120^\circ)]$ to equal $q(\theta + 240^\circ)$, the following must be satisfied:

$$\begin{aligned} c_n [1 + \cos(n \cdot 240^\circ)] &= -c_n \cos(n \cdot 240^\circ) \\ \Rightarrow c_n [1 + 2 \cos(n \cdot 240^\circ)] &= 0 \quad [1.24] \end{aligned}$$

and

$$\begin{aligned} d_n [1 + \cos(n \cdot 240^\circ)] &= -d_n \cos(n \cdot 240^\circ) \\ \Rightarrow d_n [1 + 2 \cos(n \cdot 240^\circ)] &= 0 \quad [1.25] \end{aligned}$$

The quantity $[1 + 2 \cos(n \cdot 240^\circ)]$ equals zero for $n = 1, 2, 4, 5, 7, 8, \dots$. But it is nonzero (equal to 3) for $n = 3, 6, 9, \dots$. So the only way to satisfy [1.24] and [1.25] for $n = 3, 6, 9, \dots$ is for:

$$c_n = d_n = 0 \quad ; \quad n = 3, 6, 9, \dots \quad [1.26]$$

What this means is that by assuming the winding currents satisfy [1.16] through [1.18], we are actually imposing constraints on the function $q(\theta)$.

Using [1.16] through [1.18] for the winding currents, the average torque produced over one motor revolution is given by:

$$\begin{aligned}\hat{T} &= \frac{1}{2\pi} \int_0^{2\pi} k_t i_A \cdot p(\theta) + k_t i_B \cdot p(\theta + 120^\circ) + k_t i_C \cdot p(\theta + 240^\circ) d\theta \\ &= \frac{k_t}{2\pi} \int_0^{2\pi} q(\theta) \cdot p(\theta) + q(\theta + 120^\circ) \cdot p(\theta + 120^\circ) + q(\theta + 240^\circ) \cdot p(\theta + 240^\circ) d\theta\end{aligned}\quad [1.27]$$

Replacing $p(\theta)$ with its Fourier series representation:

$$\hat{T} = \frac{k_t}{2\pi} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \cdot \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) + \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 120^\circ) + d_n \cos(n\theta + n \cdot 120^\circ) \cdot \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + b_n \cos(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 240^\circ) + d_n \cos(n\theta + n \cdot 240^\circ) \cdot \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) + b_n \cos(n\theta + n \cdot 240^\circ) \right] d\theta \quad [1.28]$$

Using some orthogonality identities (see Appendix A) and a bunch of other tricks to simplify, this becomes:

$$\begin{aligned}\hat{T} &= \frac{k_t}{2\pi} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} c_n a_n \sin^2(n\theta) + \sum_{n=1}^{\infty} c_n a_n \sin^2(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} c_n a_n \sin^2(n\theta + n \cdot 240^\circ) + \sum_{n=1}^{\infty} d_n b_n \cos^2(n\theta) + \sum_{n=1}^{\infty} d_n b_n \cos^2(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} d_n b_n \cos^2(n\theta + n \cdot 240^\circ) \right] d\theta \\ &= \frac{k_t}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} c_n a_n \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] + \sum_{n=1}^{\infty} d_n b_n \left[\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ) \right] \right\} d\theta \\ &= \frac{k_t}{2\pi} \sum_{n=1}^{\infty} \left\{ c_n a_n \int_0^{2\pi} \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] d\theta + d_n b_n \int_0^{2\pi} \left[\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ) \right] d\theta \right\} \quad [1.29] \\ &= \frac{k_t}{2\pi} \sum_{n=1}^{\infty} (3\pi c_n a_n + 3\pi d_n b_n) \\ &= \frac{3k_t}{2} \sum_{n=1}^{\infty} (c_n a_n + d_n b_n)\end{aligned}$$

Expanding the summation and imposing the necessary constraints on $p(\theta)$ described above, this becomes:

$$\hat{T} = \frac{3k_t}{2} (c_1 + c_2 a_2 + d_2 b_2 + c_4 a_4 + d_4 b_4 + c_5 a_5 + d_5 b_5 + c_7 a_7 + d_7 b_7 + \dots) \quad [1.30]$$

A huge amount of information is contained in this equation:

1. If the motor is a sinusoidal motor ($a_n = b_n = 0$; $n = 2, 3, 4, \dots$), then the average torque produced by the motor is simply proportional to the first Fourier sine coefficient of the current waveform, c_1 (i.e. the fundamental component of the winding current that is in phase with the winding curves). The type of commutation is irrelevant.

$$\hat{T}_{\text{sine_motor}} = \frac{3k_t}{2} c_1$$

2. If the winding currents are sinusoidal ($c_n = d_n = 0$; $n = 2, 3, 4, \dots$), then [1.30] reduces to the very same equation as above.

$$\hat{T}_{\text{sine_current}} = \frac{3k_t}{2} c_1$$

3. In the case of a non-sinusoidal motor and a non-sinusoidal winding current, the average torque produced can be more than, equal to, or less than what would be predicted by only accounting for the first (fundamental) Fourier component of current.

Constant-Current Trapezoidal Commutation

The current in winding A is forced to be:

$$i_A = q(\theta) = \begin{cases} +i_0 & ; \quad 30^\circ \leq \theta < 150^\circ \\ 0 & ; \quad 150^\circ \leq \theta < 210^\circ, 330^\circ \leq \theta < 390^\circ \\ -i_0 & ; \quad 210^\circ \leq \theta < 330^\circ \end{cases}$$

The Fourier sine coefficients are given by:

$$\begin{aligned} c_n &= \frac{1}{\pi} \int_0^{2\pi} i_A \sin(n\theta) d\theta = \frac{1}{\pi} \left[\int_{30^\circ}^{150^\circ} i_0 \sin(n\theta) d\theta + \int_{210^\circ}^{330^\circ} (-i_0) \sin(n\theta) d\theta \right] \\ &= \frac{i_0}{\pi} \left[\int_{30^\circ}^{150^\circ} \sin(n\theta) d\theta - \int_{210^\circ}^{330^\circ} \sin(n\theta) d\theta \right] \\ &= \frac{i_0}{\pi} \left\{ \left[-\frac{1}{n} \cos(n\theta) \right]_{30^\circ}^{150^\circ} - \left[-\frac{1}{n} \cos(n\theta) \right]_{210^\circ}^{330^\circ} \right\} \\ &= \frac{i_0}{n\pi} \left\{ \left[\cos(n\theta) \right]_{210^\circ}^{330^\circ} - \left[\cos(n\theta) \right]_{30^\circ}^{150^\circ} \right\} \\ &= \frac{i_0}{n\pi} \left\{ \left[\cos(n\theta) \right]_{\frac{7\pi}{6}}^{\frac{\pi}{6}} - \left[\cos(n\theta) \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \right\} \\ &= \frac{i_0}{n\pi} \left\{ \left[\cos\left(-n\frac{\pi}{6}\right) - \cos\left(n\frac{7\pi}{6}\right) \right] - \left[\cos\left(n\frac{5\pi}{6}\right) - \cos\left(n\frac{\pi}{6}\right) \right] \right\} \\ &= \frac{i_0}{n\pi} \left[2 \cos\left(n\frac{\pi}{6}\right) - \cos\left(n\frac{7\pi}{6}\right) - \cos\left(n\frac{5\pi}{6}\right) \right] \end{aligned}$$

This simplifies to:

$$c_n = \begin{cases} \frac{2\sqrt{3} \cdot i_0}{n\pi} & ; \quad n = 1, 11, 13, 23, 25, \dots \\ -\frac{2\sqrt{3} \cdot i_0}{n\pi} & ; \quad n = 5, 7, 17, 19, 29, 31, \dots \\ 0 & ; \quad < \text{all other } n > \end{cases}$$

The cosine coefficients are given by:

$$\begin{aligned} d_n &= \frac{1}{\pi} \int_0^{2\pi} i_A \cos(n\theta) d\theta = \frac{1}{\pi} \left[\int_{30^\circ}^{150^\circ} i_0 \cos(n\theta) d\theta + \int_{210^\circ}^{330^\circ} (-i_0) \cos(n\theta) d\theta \right] \\ &= \frac{i_0}{\pi} \left[\int_{30^\circ}^{150^\circ} \cos(n\theta) d\theta - \int_{210^\circ}^{330^\circ} \cos(n\theta) d\theta \right] = \frac{i_0}{\pi} \left\{ \left[\frac{1}{n} \sin(n\theta) \right]_{30^\circ}^{150^\circ} - \left[\frac{1}{n} \sin(n\theta) \right]_{210^\circ}^{330^\circ} \right\} \\ &= \frac{i_0}{n\pi} \left\{ \left[\sin(n\theta) \right]_{30^\circ}^{150^\circ} - \left[\sin(n\theta) \right]_{210^\circ}^{330^\circ} \right\} = \frac{i_0}{n\pi} \left\{ \left[\sin(n\theta) \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \left[\sin(n\theta) \right]_{\frac{7\pi}{6}}^{\frac{\pi}{6}} \right\} \quad [1.31] \\ &= \frac{i_0}{n\pi} \left\{ \left[\sin\left(n \frac{5\pi}{6}\right) - \sin\left(n \frac{\pi}{6}\right) \right] - \left[\sin\left(-n \frac{\pi}{6}\right) - \sin\left(n \frac{7\pi}{6}\right) \right] \right\} \\ &= \frac{i_0}{n\pi} \left[\sin\left(n \frac{5\pi}{6}\right) + \sin\left(n \frac{7\pi}{6}\right) \right] \end{aligned}$$

Noting that:

$$n \frac{5\pi}{6} = n \left(\pi - \frac{\pi}{6} \right) = n\pi - n \frac{\pi}{6}$$

and

$$n \frac{7\pi}{6} = n \left(\pi + \frac{\pi}{6} \right) = n\pi + n \frac{\pi}{6}$$

We can write:

$$\begin{aligned} d_n &= \frac{i_0}{n\pi} \left[\sin\left(n\pi - n \frac{\pi}{6}\right) + \sin\left(n\pi + n \frac{\pi}{6}\right) \right] \\ &= \frac{i_0}{n\pi} \left\{ \left[\sin(n\pi) \cos\left(-n \frac{\pi}{6}\right) + \cos(n\pi) \sin\left(-n \frac{\pi}{6}\right) \right] + \left[\sin(n\pi) \cos\left(n \frac{\pi}{6}\right) + \cos(n\pi) \sin\left(n \frac{\pi}{6}\right) \right] \right\} \\ &= \frac{i_0}{n\pi} \left[\cos(n\pi) \sin\left(-n \frac{\pi}{6}\right) + \cos(n\pi) \sin\left(n \frac{\pi}{6}\right) \right] \quad [1.32] \\ &= \frac{i_0}{n\pi} \cos(n\pi) \left[\sin\left(-n \frac{\pi}{6}\right) + \sin\left(n \frac{\pi}{6}\right) \right] \\ &= 0 \end{aligned}$$

With the Fourier coefficients of the trapezoidal winding current now known, we can plug them into [1.30] to determine the average torque for trapezoidal commutation:

$$\hat{T}_{trap} = \frac{3k_t}{2} \left(\frac{2\sqrt{3} \cdot i_0}{\pi} a_3 - \frac{2\sqrt{3} \cdot i_0}{5\pi} a_5 - \frac{2\sqrt{3} \cdot i_0}{7\pi} a_7 + \frac{2\sqrt{3} \cdot i_0}{11\pi} a_{11} + \frac{2\sqrt{3} \cdot i_0}{13\pi} a_{13} - \frac{2\sqrt{3} \cdot i_0}{17\pi} a_{17} - \frac{2\sqrt{3} \cdot i_0}{19\pi} a_{19} + \dots \right) \quad [1.33]$$

This simplifies to:

$$\hat{T}_{trap} = \frac{3\sqrt{3}k_t}{\pi} i_0 \left(1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots \right) \quad [1.34]$$

A typical “trapezoidal wound” motor has winding curves with $a_5 \approx 0.07$, $a_7 \approx 0.023$ (higher frequency coefficients fall off quickly, and their torque contributions are even further attenuated in the formula given in [1.34]). Plugging these two values into [1.34] and neglecting all higher terms, the average torque for a typical motor is:

$$\hat{T}_{trap, typical} \approx 0.983 \cdot \frac{3\sqrt{3}k_t}{\pi} i_0 \quad [1.35]$$

Determining Average Motor Torque from an (almost) Arbitrary Drive Voltage Waveform

Assume the motor controller forces the drive voltages to be of the form:

$$V_{AB} = V_A - V_B = v(\theta) \quad [1.36]$$

$$V_{BC} = V_B - V_C = v(\theta + 120^\circ) \quad [1.37]$$

$$V_{CA} = V_C - V_A = v(\theta + 240^\circ) \quad [1.38]$$

Where:

$$v(\theta) = \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) \quad [1.39]$$

By the way, note that the negative of the summation of [1.36] and [1.37] yields:

$$V_C - V_A = -[v(\theta) + v(\theta + 120^\circ)] \quad [1.40]$$

Comparing this with [1.38], we realize that in order for [1.36] through [1.38] to be satisfied, the following relation must be true:

$$v(\theta + 240^\circ) = -[v(\theta) + v(\theta + 120^\circ)] \quad [1.41]$$

This is the same scenario we had while deriving torque due to an arbitrary current waveform, and borrowing from that section, we know that:

$$g_n = h_n = 0 ; n = 3, 6, 9, \dots \quad [1.42]$$

As was the case before, this means that by assuming the drive voltages satisfy [1.36] through [1.38], we are actually imposing constraints on the function $v(\theta)$.

Performing KVL from the A phase to the B phase (neglecting the winding inductance for now):

$$V_{AB} = i_A R + \varepsilon_A - \varepsilon_B - i_B R \quad [1.43]$$

And from phase B to C:

$$V_{BC} = i_B R + \varepsilon_B - \varepsilon_C - i_C R \quad [1.44]$$

Since the drive voltages and winding EMF's are known, we rearrange these as follows:

$$i_A - i_B = \frac{V_{AB} - \varepsilon_A + \varepsilon_B}{R} \quad [1.45]$$

$$i_B - i_C = \frac{V_{BC} - \varepsilon_B + \varepsilon_C}{R} \quad [1.46]$$

These two equations combined with conservation of current,

$$i_A + i_B + i_C = 0 \quad [1.47]$$

form a system of three equations with three unknowns (the winding currents). Solving the system yields:

$$i_A = \frac{2V_{AB} + V_{BC} - 2\varepsilon_A + \varepsilon_B + \varepsilon_C}{3R} \quad [1.48]$$

$$i_B = \frac{-V_{AB} + V_{BC} + \varepsilon_A - 2\varepsilon_B + \varepsilon_C}{3R} \quad [1.49]$$

$$i_C = \frac{-V_{AB} - 2V_{BC} + \varepsilon_A + \varepsilon_B - 2\varepsilon_C}{3R} \quad [1.50]$$

Focusing now on i_A , we plug in for the winding drive voltages and winding EMF's:

$$i_A = \frac{2v(\theta) + v(\theta + 120^\circ) + k_v \dot{\theta} \left[-2p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ) \right]}{3R} \quad [1.51]$$

Replacing the functions $v(\theta)$ and $p(\theta)$ with their Fourier series representations:

$$\begin{aligned} i_A &= \frac{1}{3R} \left\{ 2 \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) + \sum_{n=1}^{\infty} g_n \sin(n\theta + n \cdot 120^\circ) + h_n \cos(n\theta + n \cdot 120^\circ) + \right. \\ &\quad \left. k_v \dot{\theta} \left[-2 \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) + \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + b_n \cos(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) + b_n \cos(n\theta + n \cdot 240^\circ) \right] \right\} \\ i_A &= \frac{1}{3R} \left\{ 2 \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) + \sum_{n=1}^{\infty} g_n \sin(n\theta + n \cdot 120^\circ) + h_n \cos(n\theta + n \cdot 120^\circ) + \right. \\ &\quad \left. k_v \dot{\theta} \left[-2 \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) + \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + b_n \cos(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) + b_n \cos(n\theta + n \cdot 240^\circ) \right] \right\} \\ &= \frac{1}{3R} \left\{ \sum_{n=1}^{\infty} g_n \left[2 \sin(n\theta) + \sin(n\theta + n \cdot 120^\circ) \right] + h_n \left[2 \cos(n\theta) + \cos(n\theta + n \cdot 120^\circ) \right] + \right. \\ &\quad \left. k_v \dot{\theta} \left[\sum_{n=1}^{\infty} a_n \left[-2 \sin(n\theta) + \sin(n\theta + n \cdot 120^\circ) + \sin(n\theta + n \cdot 240^\circ) \right] + b_n \left[-2 \cos(n\theta) + \cos(n\theta + n \cdot 120^\circ) + \cos(n\theta + n \cdot 240^\circ) \right] \right] \right\} \\ &= \frac{1}{3R} \left\{ \sum_{n=1}^{\infty} g_n \left[2 \sin(n\theta) + \sin(n\theta) \cos(n \cdot 120^\circ) + \cos(n\theta) \sin(n \cdot 120^\circ) \right] + h_n \left[2 \cos(n\theta) + \cos(n\theta) \cos(n \cdot 120^\circ) - \sin(n\theta) \sin(n \cdot 120^\circ) \right] + \right. \\ &\quad \left. k_v \dot{\theta} \sum_{n=1}^{\infty} \left\{ a_n \left[-2 \sin(n\theta) + \sin(n\theta) \cos(n \cdot 120^\circ) + \cos(n\theta) \sin(n \cdot 120^\circ) + \sin(n\theta) \cos(n \cdot 240^\circ) + \cos(n\theta) \sin(n \cdot 240^\circ) \right] + \right. \right. \\ &\quad \left. \left. b_n \left[-2 \cos(n\theta) + \cos(n\theta) \cos(n \cdot 120^\circ) - \sin(n\theta) \sin(n \cdot 120^\circ) + \cos(n\theta) \cos(n \cdot 240^\circ) - \sin(n\theta) \sin(n \cdot 240^\circ) \right] \right\} \right\} \\ &= \frac{1}{3R} \left\{ \sum_{n=1}^{\infty} \left[2g_n + g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ) \right] \sin(n\theta) + \left[g_n \sin(n \cdot 120^\circ) + 2h_n + h_n \cos(n \cdot 120^\circ) \right] \cos(n\theta) + \right. \\ &\quad \left. k_v \dot{\theta} \sum_{n=1}^{\infty} \left\{ \left[-2a_n + a_n \cos(n \cdot 120^\circ) + a_n \cos(n \cdot 240^\circ) - b_n \sin(n \cdot 120^\circ) - b_n \sin(n \cdot 240^\circ) \right] \sin(n\theta) + \right. \right. \\ &\quad \left. \left. \left[a_n \sin(n \cdot 120^\circ) + a_n \sin(n \cdot 240^\circ) - 2b_n + b_n \cos(n \cdot 120^\circ) + b_n \cos(n \cdot 240^\circ) \right] \cos(n\theta) \right\} \right\} \\ &= \frac{1}{3R} \sum_{n=1}^{\infty} \left\{ \left[\left[2g_n + g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ) \right] + k_v \dot{\theta} \left[-2a_n + a_n \cos(n \cdot 120^\circ) + a_n \cos(n \cdot 240^\circ) - b_n \sin(n \cdot 120^\circ) - b_n \sin(n \cdot 240^\circ) \right] \right] \sin(n\theta) + \right. \\ &\quad \left. \left[\left[g_n \sin(n \cdot 120^\circ) + 2h_n + h_n \cos(n \cdot 120^\circ) \right] + k_v \dot{\theta} \left[a_n \sin(n \cdot 120^\circ) + a_n \sin(n \cdot 240^\circ) - 2b_n + b_n \cos(n \cdot 120^\circ) + b_n \cos(n \cdot 240^\circ) \right] \right] \cos(n\theta) \right\} \end{aligned} \quad [1.52]$$

Now we have the winding current written as a Fourier series. Its Fourier coefficients are, by inspection:

$$\begin{aligned} c_n &= \frac{1}{3R} \left\{ \left[2g_n + g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ) \right] + k_v \dot{\theta} \left[-2a_n + a_n \cos(n \cdot 120^\circ) + a_n \cos(n \cdot 240^\circ) - b_n \sin(n \cdot 120^\circ) - b_n \sin(n \cdot 240^\circ) \right] \right\} \quad [1.53] \\ &= \frac{1}{3R} \left\{ \left[g_n (2 + \cos(n \cdot 120^\circ)) - h_n \sin(n \cdot 120^\circ) \right] + k_v \dot{\theta} \left[a_n (-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)) - b_n (\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ)) \right] \right\} \end{aligned}$$

$$\begin{aligned} d_n &= \frac{1}{3R} \left\{ \left[g_n \sin(n \cdot 120^\circ) + 2h_n + h_n \cos(n \cdot 120^\circ) \right] + k_v \dot{\theta} \left[a_n \sin(n \cdot 120^\circ) + a_n \sin(n \cdot 240^\circ) - 2b_n + b_n \cos(n \cdot 120^\circ) + b_n \cos(n \cdot 240^\circ) \right] \right\} \quad [1.54] \\ &= \frac{1}{3R} \left\{ \left[g_n \sin(n \cdot 120^\circ) + h_n (2 + \cos(n \cdot 120^\circ)) \right] + k_v \dot{\theta} \left[a_n (\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ)) + b_n (-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)) \right] \right\} \end{aligned}$$

Using [1.116] from Appendix A, this simplifies to:

$$c_n = \frac{1}{3R} \left\{ \left[g_n (2 + \cos(n \cdot 120^\circ)) - h_n \sin(n \cdot 120^\circ) \right] + k_v \dot{\theta} a_n \left[-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ) \right] \right\} \quad [1.55]$$

$$d_n = \frac{1}{3R} \left\{ \left[g_n \sin(n \cdot 120^\circ) + h_n (2 + \cos(n \cdot 120^\circ)) \right] + k_v \dot{b}_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \right\} \quad [1.56]$$

And using :

$$2 + \cos(n \cdot 120^\circ) = \begin{cases} \frac{3}{2} & ; \quad n = 1, 2, 4, 5, 7, 8, \dots \\ 3 & ; \quad n = 3, 6, 9, \dots \end{cases}$$

Combined with the fact that we already know $g_n = h_n = 0$; $n = 3, 6, 9, \dots$, we can write:

$$c_n = \frac{1}{3R} \left\{ \left[\frac{3}{2} g_n - h_n \sin(n \cdot 120^\circ) \right] + k_v \dot{a}_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \right\} \quad [1.57]$$

$$d_n = \frac{1}{3R} \left\{ \left[g_n \sin(n \cdot 120^\circ) + \frac{3}{2} h_n \right] + k_v \dot{b}_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \right\} \quad [1.58]$$

Also using:

$$\cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ) = \begin{cases} -1 & ; \quad n = 1, 2, 4, 5, 7, 8, \dots \\ 2 & ; \quad n = 3, 6, 9, \dots \end{cases} \quad [1.59]$$

$$\sin(n \cdot 120^\circ) = \begin{cases} \frac{\sqrt{3}}{2} & ; \quad n = 1, 4, 7, \dots \\ -\frac{\sqrt{3}}{2} & ; \quad n = 2, 5, 8, \dots \\ 0 & ; \quad n = 3, 6, 9, \dots \end{cases} \quad [1.60]$$

We can write:

$$c_n = \begin{cases} \frac{1}{3R} \left(\frac{3}{2} g_n - \frac{\sqrt{3}}{2} h_n - 3k_v \dot{a}_n \right) & ; \quad n = 1, 4, 7, \dots \\ \frac{1}{3R} \left(\frac{3}{2} g_n + \frac{\sqrt{3}}{2} h_n - 3k_v \dot{a}_n \right) & ; \quad n = 2, 5, 8, \dots \\ 0 & ; \quad n = 3, 6, 9, \dots \end{cases} = \begin{cases} \frac{1}{R} \left(\frac{1}{2} g_n - \frac{1}{2\sqrt{3}} h_n - k_v \dot{a}_n \right) & ; \quad n = 1, 4, 7, \dots \\ \frac{1}{R} \left(\frac{1}{2} g_n + \frac{1}{2\sqrt{3}} h_n - k_v \dot{a}_n \right) & ; \quad n = 2, 5, 8, \dots \\ 0 & ; \quad n = 3, 6, 9, \dots \end{cases} \quad [1.61]$$

$$d_n = \begin{cases} \frac{1}{3R} \left(\frac{\sqrt{3}}{2} g_n + \frac{3}{2} h_n - 3k_v \dot{b}_n \right) & ; \quad n = 1, 4, 7, \dots \\ \frac{1}{3R} \left(-\frac{\sqrt{3}}{2} g_n + \frac{3}{2} h_n - 3k_v \dot{b}_n \right) & ; \quad n = 2, 5, 8, \dots \\ 0 & ; \quad n = 3, 6, 9, \dots \end{cases} = \begin{cases} \frac{1}{R} \left(\frac{1}{2\sqrt{3}} g_n + \frac{1}{2} h_n - k_v \dot{b}_n \right) & ; \quad n = 1, 4, 7, \dots \\ \frac{1}{R} \left(-\frac{1}{2\sqrt{3}} g_n + \frac{1}{2} h_n - k_v \dot{b}_n \right) & ; \quad n = 2, 5, 8, \dots \\ 0 & ; \quad n = 3, 6, 9, \dots \end{cases} \quad [1.62]$$

With the Fourier coefficients of i_A known, we can use [1.30] to determine average motor torque.

(Notice that $c_n = d_n = 0$; $n = 3, 6, 9, \dots$, which is required for [1.30] to be valid.)

$$\begin{aligned}
\hat{T} &= \frac{3k_t}{2} \left(\frac{1}{R} \left(\frac{1}{2} g_1 - \frac{1}{2\sqrt{3}} h_1 - k_v \dot{\theta} \right) + \frac{1}{R} \left(\frac{1}{2} g_2 + \frac{1}{2\sqrt{3}} h_2 - k_v \dot{\theta} a_2 \right) a_2 + \frac{1}{R} \left(-\frac{1}{2\sqrt{3}} g_2 + \frac{1}{2} h_2 - k_v \dot{\theta} b_2 \right) b_2 + \frac{1}{R} \left(\frac{1}{2} g_4 - \frac{1}{2\sqrt{3}} h_4 - k_v \dot{\theta} a_4 \right) a_4 + \right. \\
&\quad \left. \frac{1}{R} \left(\frac{1}{2\sqrt{3}} g_4 + \frac{1}{2} h_4 - k_v \dot{\theta} b_4 \right) b_4 + \frac{1}{R} \left(\frac{1}{2} g_5 + \frac{1}{2\sqrt{3}} h_5 - k_v \dot{\theta} a_5 \right) a_5 + \frac{1}{R} \left(-\frac{1}{2\sqrt{3}} g_5 + \frac{1}{2} h_5 - k_v \dot{\theta} b_5 \right) b_5 \dots \right) \\
&= \frac{3k_t}{2R} \left(\left(\frac{1}{2} g_1 - \frac{1}{2\sqrt{3}} h_1 - k_v \dot{\theta} \right) + \left(\frac{1}{2} g_2 + \frac{1}{2\sqrt{3}} h_2 - k_v \dot{\theta} a_2 \right) a_2 + \left(-\frac{1}{2\sqrt{3}} g_2 + \frac{1}{2} h_2 - k_v \dot{\theta} b_2 \right) b_2 + \left(\frac{1}{2} g_4 - \frac{1}{2\sqrt{3}} h_4 - k_v \dot{\theta} a_4 \right) a_4 + \right. \\
&\quad \left. \left(\frac{1}{2\sqrt{3}} g_4 + \frac{1}{2} h_4 - k_v \dot{\theta} b_4 \right) b_4 + \left(\frac{1}{2} g_5 + \frac{1}{2\sqrt{3}} h_5 - k_v \dot{\theta} a_5 \right) a_5 + \left(-\frac{1}{2\sqrt{3}} g_5 + \frac{1}{2} h_5 - k_v \dot{\theta} b_5 \right) b_5 + \dots \right) \\
&= \frac{3k_t}{2R} \left(\frac{1}{2} g_1 - \frac{1}{2\sqrt{3}} h_1 - k_v \dot{\theta} + \frac{1}{2} g_2 a_2 + \frac{1}{2\sqrt{3}} h_2 a_2 - k_v \dot{\theta} a_2^2 - \frac{1}{2\sqrt{3}} g_2 b_2 + \frac{1}{2} h_2 b_2 - k_v \dot{\theta} b_2^2 + \frac{1}{2} g_4 a_4 - \frac{1}{2\sqrt{3}} h_4 a_4 - k_v \dot{\theta} a_4^2 + \right. \\
&\quad \left. \frac{1}{2\sqrt{3}} g_4 b_4 + \frac{1}{2} h_4 b_4 - k_v \dot{\theta} b_4^2 + \frac{1}{2} g_5 a_5 + \frac{1}{2\sqrt{3}} h_5 a_5 - k_v \dot{\theta} a_5^2 - \frac{1}{2\sqrt{3}} g_5 b_5 + \frac{1}{2} h_5 b_5 - k_v \dot{\theta} b_5^2 + \dots \right) \\
&= \frac{3k_t}{2R} \left(\frac{1}{2} (g_1 + g_2 a_2 + h_2 b_2 + g_4 a_4 + h_4 b_4 + g_5 a_5 + h_5 b_5 + \dots) + \right. \\
&\quad \left. \frac{1}{2\sqrt{3}} (-h_1 + h_2 a_2 - g_2 b_2 - h_4 a_4 + g_4 b_4 + h_5 a_5 - g_5 b_5 + \dots) - k_v \dot{\theta} (1 + a_2^2 + b_2^2 + a_4^2 + b_4^2 + a_5^2 + b_5^2 + \dots) \right)
\end{aligned}$$

Rewriting in its final form, we have:

$$\boxed{\hat{T} = \frac{3k_t}{2R} \left(\frac{1}{2} (g_1 + g_2 a_2 + h_2 b_2 + g_4 a_4 + h_4 b_4 + g_5 a_5 + h_5 b_5 + \dots) + \frac{1}{2\sqrt{3}} (-h_1 + h_2 a_2 - g_2 b_2 - h_4 a_4 + g_4 b_4 + h_5 a_5 - g_5 b_5 + \dots) - k_v \dot{\theta} (1 + a_2^2 + b_2^2 + a_4^2 + b_4^2 + a_5^2 + b_5^2 + \dots) \right)} \quad [1.63]$$

for a motor with fully symmetric winding curves (i.e. $p(\theta)$ is odd and symmetric about $\pi/2$), this further simplifies to:

$$\hat{T}_{\text{sym_motor}} = \frac{3k_t}{2R} \left[\frac{1}{2} (g_1 + g_5 a_5 + \dots) + \frac{1}{2\sqrt{3}} (-h_1 + h_5 a_5 + \dots) - k_v \dot{\theta} (1 + a_5^2 + \dots) \right] \quad [1.64]$$

Note that this analysis has neglected the effects of the winding inductance. Appendix D contains a more complex analysis for arbitrary drive voltages, taking into account the winding inductance.

Pseudo-Sinusoidal Commutation

The MSL motor controller does not implement true sinusoidal current commutation, in that it does not force the winding currents to be sinusoidal. Instead, it drives the motor with sinusoidal voltages that would result in sinusoidal winding currents in the case of a truly sinusoidal motor. In the case of a non-sinusoidal motor, the actual winding currents are distorted.

First, let's derive the voltages used to drive the motor. (We know they will be sinusoidal, but their phases are not obvious.) Performing KVL from the A phase to the B phase:

$$V_A - V_B = i_A R + \varepsilon_A - \varepsilon_B - i_B R \quad [1.65]$$

Assuming a sinusoidal motor, the winding EMF's are given by [1.1] through [1.3]. And since we are trying to solve for the voltages needed to make the winding currents sinusoidal, we assume sinusoidal forms for them:

$$\begin{aligned}
V_A - V_B &= I \sin(\theta) R + k_v \dot{\theta} \sin(\theta) - k_v \dot{\theta} \sin(\theta + 120^\circ) - I \sin(\theta + 120^\circ) R \\
&= RI \left[\sin(\theta) - \sin(120^\circ) \right] + k_v \dot{\theta} \left[\sin(\theta) - \sin(\theta + 120^\circ) \right] \\
&= (RI + k_v \dot{\theta}) \left[\sin(\theta) - \sin(120^\circ) \right]
\end{aligned} \quad [1.66]$$

Noting that:

$$\sin(\theta) - \sin(\theta + 120^\circ) = \sqrt{3} \sin(\theta - 30^\circ) \quad [1.67]$$

We can write:

$$V_A - V_B = \sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta - 30^\circ) \quad [1.68]$$

Similarly, we find that:

$$V_A - V_C = \sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta + 30^\circ) \quad [1.69]$$

We can arbitrarily decide to let V_A be our ground reference:

$$V_{A,\text{ sine}} = 0, \quad [1.70]$$

which means:

$$V_{B,\text{ sine}} = -\sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta - 30^\circ) \quad [1.71]$$

$$V_{C,\text{ sine}} = -\sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta + 30^\circ) \quad [1.72]$$

The amplitude of these sinusoids represents the amplitude required to drive a particular sinusoidal motor at a given speed. In practice, the drive voltage amplitude is decided by the motor controller (either specified by the user, or by some other control loop). Therefore, we'll rewrite the voltages as simply:

$$V_{B,\text{ sine}} = -\sqrt{3} V_0 \sin(\theta - 30^\circ) \quad [1.73]$$

$$V_{C,\text{ sine}} = -\sqrt{3} V_0 \sin(\theta + 30^\circ) \quad [1.74]$$

The reason for keeping the $\sqrt{3}$ will become apparent down the road.

With the sine drive voltages now known (specifically their phases), we turn back to a non-sinusoidal motor. We want to determine what the winding currents will look like, since we know they will not be purely sinusoidal. Rewriting [1.65], we get:

$$V_{B,\text{ sine}} = \mathcal{E}_B - \mathcal{E}_A + R(i_B - i_A) \Rightarrow i_B - i_A = \frac{V_{B,\text{ sine}} - \mathcal{E}_B + \mathcal{E}_A}{R} \quad [1.75]$$

Doing the same (KVL) from phase C to A:

$$V_{C,\text{ sine}} = \mathcal{E}_C - \mathcal{E}_A + R(i_C - i_A) \Rightarrow i_C - i_A = \frac{V_{C,\text{ sine}} - \mathcal{E}_C + \mathcal{E}_A}{R} \quad [1.76]$$

These two relations along with the obvious conservation of current:

$$i_A + i_B + i_C = 0 \quad [1.77]$$

yield a system of three equations in three unknowns (the three winding currents). The solution to the system is:

$$i_A = \frac{-V_{B,\text{ sine}} - V_{C,\text{ sine}} - 2\mathcal{E}_A + \mathcal{E}_B + \mathcal{E}_C}{3R} \quad [1.78]$$

$$i_B = \frac{2V_{B,\text{ sine}} - V_{C,\text{ sine}} + \mathcal{E}_A - 2\mathcal{E}_B + \mathcal{E}_C}{3R} \quad [1.79]$$

$$i_C = \frac{-V_{B,\text{ sine}} + 2V_{C,\text{ sine}} + \mathcal{E}_A + \mathcal{E}_B - 2\mathcal{E}_C}{3R} \quad [1.80]$$

We now have the winding currents in terms of the drive voltages (given by [1.73] and [1.74]) and the winding EMF's (given in general by [1.7] through [1.9]). Focusing now on i_A , we can plug the drive voltages and winding EMF's into [1.78] to get:

$$i_A = \frac{\sqrt{3}V_0 \sin(\theta - 30^\circ) + \sqrt{3}V_0 \sin(\theta + 30^\circ) - 2k_v \dot{\theta} p(\theta) + k_v \dot{\theta} p(\theta + 120^\circ) + k_v \dot{\theta} p(\theta + 240^\circ)}{3R} \quad [1.81]$$

$$= \frac{\sqrt{3}V_0 [\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ)] + k_v \dot{\theta} [-2p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ)]}{3R}$$

Using [1.113] from Appendix A and writing $p(\theta)$ in terms of its Fourier series:

$$i_A = \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \left[\begin{aligned} & -2 \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) + \\ & \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + b_n \cos(n\theta + n \cdot 120^\circ) + \\ & \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) + b_n \cos(n\theta + n \cdot 240^\circ) \end{aligned} \right] \quad [1.82]$$

$$= \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} \left\{ \begin{aligned} & a_n [-2 \sin(n\theta) + \sin(n\theta + n \cdot 120^\circ) + \sin(n\theta + n \cdot 240^\circ)] + \\ & b_n [-2 \cos(n\theta) + \cos(n\theta + n \cdot 120^\circ) + \cos(n\theta + n \cdot 240^\circ)] \end{aligned} \right\}$$

Now making use of [1.109]:

$$i_A = \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} \left\{ \begin{aligned} & a_n [-2 \sin(n\theta) + \sin(n\theta) \cos(n \cdot 120^\circ) + \sin(n \cdot 120^\circ) \cos(n\theta) + \sin(n\theta) \cos(n \cdot 240^\circ) + \sin(n \cdot 240^\circ) \cos(n\theta)] + \\ & b_n [-2 \cos(n\theta) + \cos(n\theta) \cos(n \cdot 120^\circ) - \sin(n\theta) \sin(n \cdot 120^\circ) + \cos(n\theta) \cos(n \cdot 240^\circ) - \sin(n\theta) \sin(n \cdot 240^\circ)] \end{aligned} \right\}$$

Grouping the $\sin(n\theta)$ and $\cos(n\theta)$ terms:

$$i_A = \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} \left\{ \begin{aligned} & a_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \sin(n\theta) + a_n [\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ)] \cos(n\theta) + \\ & b_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \cos(n\theta) + b_n [-\sin(n \cdot 120^\circ) - \sin(n \cdot 240^\circ)] \sin(n\theta) \end{aligned} \right\} \quad [1.83]$$

$$= \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} \left\{ \begin{aligned} & \{a_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] + b_n [-\sin(n \cdot 120^\circ) - \sin(n \cdot 240^\circ)]\} \sin(n\theta) + \\ & \{a_n [\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ)] + b_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)]\} \cos(n\theta) \end{aligned} \right\}$$

Noting that:

$$\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ) = 0 ; \quad \forall n \in \mathbb{Z} \quad [1.84]$$

This simplifies [1.83] to:

$$i_A = \frac{V_0}{R} \sin(\theta) + \frac{k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} \left(\begin{aligned} & a_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \sin(n\theta) + \\ & b_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] \cos(n\theta) \end{aligned} \right) \quad [1.85]$$

In order to make use of [1.30] to determine average motor torque, we must find the Fourier coefficients of i_A . In general, this requires us to perform the Fourier coefficient integral, but in this case, [1.85] is already in the form of a Fourier series, so the coefficients can be written by inspection:

$$c_n = \begin{cases} \frac{V_0}{R} + \frac{k_v \dot{\theta}}{3R} [-2 + \cos(120^\circ) + \cos(240^\circ)] & ; \quad n = 1 \\ \frac{k_v \dot{\theta}}{3R} a_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] & ; \quad n = 2, 3, \dots \end{cases} \quad [1.86]$$

and

$$d_n = \frac{k_v \dot{\theta}}{3R} b_n [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] ; \quad n = 1, 2, 3, \dots \quad [1.87]$$

Taking note of the following pattern:

n	$-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)$
1	-3
2	-3
3	0
4	-3
5	-3
6	0
7	-3
$\vdots$	$\vdots$

we can clean up [1.86] and [1.87]:

$$c_n = \begin{cases} \frac{V_0 - k_v \dot{\theta}}{R} & ; \quad n=1 \\ -\frac{k_v \dot{\theta}}{R} a_n & ; \quad n=2,4,5,7,8,\dots \\ 0 & ; \quad n=3,6,9,12,\dots \end{cases} \quad [1.88]$$

and

$$d_n = \begin{cases} -\frac{k_v \dot{\theta}}{R} b_n & ; \quad n=1,2,4,5,7,8,\dots \\ 0 & ; \quad n=3,6,9,12,\dots \end{cases} \quad [1.89]$$

We can now plug these Fourier coefficients into [1.30] to determine average motor torque:

$$\hat{T} = \frac{3k_t}{2} \left(\frac{V_0 - k_v \dot{\theta}}{R} - \frac{k_v \dot{\theta}}{R} a_2^2 - \frac{k_v \dot{\theta}}{R} b_2^2 - \frac{k_v \dot{\theta}}{R} a_4^2 - \frac{k_v \dot{\theta}}{R} b_4^2 - \frac{k_v \dot{\theta}}{R} a_5^2 - \frac{k_v \dot{\theta}}{R} b_5^2 - \frac{k_v \dot{\theta}}{R} a_7^2 - \frac{k_v \dot{\theta}}{R} b_7^2 - \dots \right) \quad [1.90]$$

Rearranging slightly, the average motor torque is given by:

$$\hat{T} = \frac{3k_t}{2} \left[\left(\frac{V_0 - k_v \dot{\theta}}{R} \right) - \frac{k_v \dot{\theta}}{R} (a_2^2 + b_2^2 + a_4^2 + b_4^2 + a_5^2 + b_5^2 + a_7^2 + b_7^2 + \dots) \right] \quad [1.91]$$

In the special case where $p(\theta)$ is odd, this simplifies to:

$$\hat{T} \Big|_{p(-\theta)=-p(\theta)} = \frac{3k_t}{2} \left[\left(\frac{V_0 - k_v \dot{\theta}}{R} \right) - \frac{k_v \dot{\theta}}{R} (a_2^2 + a_4^2 + a_5^2 + a_7^2 + a_8^2 + a_{10}^2 + a_{11}^2 \dots) \right] \quad [1.92]$$

And if $p(\theta)$ is further constrained to be symmetric about $\frac{\pi}{2}$, [1.92] simplifies to just:

$$\hat{T} \Big|_{\substack{p(-\theta)=-p(\theta) \\ p(\theta)=p(\pi-\theta)}} = \frac{3k_t}{2} \left[\left(\frac{V_0 - k_v \dot{\theta}}{R} \right) - \frac{k_v \dot{\theta}}{R} (a_5^2 + a_7^2 + a_{11}^2 + a_{13}^2 + a_{17}^2 + a_{19}^2 + \dots) \right] \quad [1.93]$$

Deriving Winding Dissipation from an Arbitrary Winding Current Waveform

If winding current data is available, the average power dissipation in the winding resistance is given by:

$$\begin{aligned}
 \hat{P}_{diss,w} &= \frac{1}{2\pi} \int_0^{2\pi} R(i_A^2 + i_B^2 + i_C^2) d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} q(\theta)^2 + q(\theta + 120^\circ)^2 + q(\theta + 240^\circ)^2 d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \left[\sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \right]^2 + \right. \\
 &\quad \left[\sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 120^\circ) + d_n \cos(n\theta + n \cdot 120^\circ) \right]^2 + \\
 &\quad \left. \left[\sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 240^\circ) + d_n \cos(n\theta + n \cdot 240^\circ) \right]^2 \right\} d\theta
 \end{aligned} \tag{1.94}$$

Again we use orthogonality to simplify this to:

$$\begin{aligned}
 \hat{P}_{diss,w} &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta) + d_n^2 \cos^2(n\theta) + \right. \\
 &\quad \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta + n \cdot 120^\circ) + d_n^2 \cos^2(n\theta + n \cdot 120^\circ) + \\
 &\quad \left. \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta + n \cdot 240^\circ) + d_n^2 \cos^2(n\theta + n \cdot 240^\circ) \right\} d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} c_n^2 [\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ)] + \right. \\
 &\quad \left. \sum_{n=1}^{\infty} d_n^2 [\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ)] \right\} d\theta \\
 &= \frac{R}{2\pi} \sum_{n=1}^{\infty} \left\{ c_n^2 \int_0^{2\pi} [\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ)] d\theta + \right. \\
 &\quad \left. d_n^2 \int_0^{2\pi} [\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ)] d\theta \right\} \\
 &= \frac{R}{2\pi} \sum_{n=1}^{\infty} (c_n^2 \cdot 3\pi + d_n^2 \cdot 3\pi) = \frac{3R}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2)
 \end{aligned} \tag{1.95}$$

Rewriting this in its final form:

$$\boxed{\hat{P}_{diss,w} = \frac{3R}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2)} \tag{1.96}$$

Recall (as can be shown using similar methods as above) that the sum of the squares of the Fourier coefficients is related to a function's RMS value by:

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \cdot f_{RMS}^2 \quad ; \quad \text{for } f(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) \tag{1.97}$$

Rewriting the average dissipation in terms of the RMS winding current:

$$\boxed{\hat{P}_{diss,w} = 3 \cdot q_{RMS}^2 \cdot R} \tag{1.98}$$

Now that I'm here, this should have been obvious – it's the simple Joule heating relationship for AC current (the factor of three is there to account for all three windings).

Estimating Efficiency from Arbitrary Winding Current, Torque, and Speed Waveforms

The motor efficiency is defined by:

$$\eta \equiv \frac{P_{mech_output}}{P_{input}} \quad [1.99]$$

where P_{mech_output} denotes the mechanical output (shaft) power from the motor, and P_{input} denotes the total electrical power into the motor. All the input power must go somewhere – it can be stored (in the motor), dissipated (as Joule heating, hysteresis loss, viscous drag, etc.) , and/or transmitted as output power:

$$P_{input} = \dot{E}_{motor} + P_{diss} + P_{mech_output} \quad [1.100]$$

What we're really after is average efficiency more so than instantaneous efficiency. we'll define it as (note this is not the same thing as mathematically averaging [1.99]):

$$\hat{\eta} \equiv \frac{\hat{P}_{mech_output}}{\hat{P}_{input}} \quad [1.101]$$

Realizing that the average rate of energy stored in the motor must be zero at steady state, we can plug the average of [1.100] into [1.101] to eliminate $\hat{P}_{input}$ (mainly because it's difficult to measure):

$$\hat{\eta} \equiv \frac{\hat{P}_{mech_output}}{\hat{P}_{diss} + \hat{P}_{mech_output}} \quad [1.102]$$

Now we make an assumption that the average power dissipation is largely dominated by Joule heating in the windings. In other words, we neglect other losses such as hysteresis loss, viscous drag, etc. In practice for most motors, this has shown to be true. In this case, the dissipation is give by equation [1.98].

Note that the mechanical output power is given by:

$$P_{mech_output} = T \cdot \dot{\theta} \Rightarrow \hat{P}_{mech_output} \approx \hat{T} \cdot \hat{\dot{\theta}} \quad [1.103]$$

If torque and speed measurement data is available, the motor efficiency is given by:

$$\hat{\eta} \equiv \frac{\hat{T} \cdot \hat{\dot{\theta}}}{3 \cdot q_{RMS}^2 \cdot R + \hat{T} \cdot \hat{\dot{\theta}}} \quad [1.104]$$

Now we will examine the effect winding curve distortion has on a motor's efficiency while being operated with pseudo-sinusoidal commutation. To do this, we need to determine the RMS winding current. Making use of [1.97] and the fact that we've already derived the Fourier coefficients of winding current for this commutation type (equations [1.88] and [1.89]), we can directly write an expression for the square of the RMS winding current:

$$\begin{aligned}
q_{RMS}^2 &= \frac{1}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2) \\
&= \frac{1}{2} \left[\left(\frac{V_0 - k_v \dot{\theta}}{R} \right)^2 + \left(\frac{k_v \dot{\theta}}{R} b_1 \right)^2 + \left(\frac{k_v \dot{\theta}}{R} a_2 \right)^2 + \left(\frac{k_v \dot{\theta}}{R} b_2 \right)^2 + \left(\frac{k_v \dot{\theta}}{R} a_4 \right)^2 + \left(\frac{k_v \dot{\theta}}{R} b_4 \right)^2 + \dots \right] \quad [1.105] \\
&= \frac{1}{2} \left[\left(\frac{V_0 - k_v \dot{\theta}}{R} \right)^2 + \left(\frac{k_v \dot{\theta}}{R} \right)^2 (a_2^2 + b_2^2 + a_4^2 + b_4^2 + a_5^2 + b_5^2 + a_7^2 + b_7^2 + a_8^2 + b_8^2 + a_{10}^2 + b_{10}^2 + \dots) \right]
\end{aligned}$$

This tells us that the first term in the denominator of [1.104] is at its lowest when $p(\theta)$ is sinusoidal.

Any non-zero Fourier coefficient above the fundamental frequency tends to increase the winding dissipation. On the flip side, [1.91] tells us the average torque, $\hat{T}$, is at its highest when $p(\theta)$ is sinusoidal. Any non-zero Fourier coefficient above the fundamental frequency tends to decrease the average motor torque. What this all means is the more distorted the winding curves are for a motor, the lower its efficiency will be when operated using pseudo-sinusoidal commutation.

Appendices

Appendix A – Useful Identities

The following identities are used in many places in the above analysis. These identities are referred to collectively as orthogonality identities.

$$\int_{\theta_0}^{\theta_0+2\pi} \sin(m\theta) \sin(n\theta) d\theta = \int_{\theta_0}^{\theta_0+2\pi} \cos(m\theta) \cos(n\theta) d\theta = 0 \quad ; \quad n \neq m \quad \forall \quad n, m \in \mathbb{Z} \quad [1.106]$$

$$\int_{\theta_0}^{\theta_0+2\pi} \sin(m\theta) \cos(n\theta) d\theta = 0 \quad ; \quad \forall \quad n, m \in \mathbb{Z} \quad [1.107]$$

$$\int_{\theta_0}^{\theta_0+2\pi} \sin^2(n\theta + b) d\theta = \int_{\theta_0}^{\theta_0+2\pi} \cos^2(n\theta + b) d\theta = \pi \quad ; \quad \forall \quad n \in \mathbb{Z}, b \in \mathbb{R} \quad [1.108]$$

Below are some other useful trigonometric identities:

$$\sin(a + b) = \cos(b) \sin(a) + \sin(b) \cos(a) \quad ; \quad \forall \quad a, b \in \mathbb{R} \quad [1.109]$$

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \quad ; \quad \forall \quad a, b \in \mathbb{R}$$

$$\sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) = \frac{3}{2} \quad [1.110]$$

$$\sin(\theta) - \sin(\theta + 120^\circ) = \sqrt{3} \sin(\theta - 30^\circ) \quad [1.111]$$

$$\sin(\theta) - \sin(\theta + 240^\circ) = \sqrt{3} \sin(\theta + 30^\circ) \quad [1.112]$$

$$\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta) \quad [1.113]$$

$$-2 \sin(\theta - 30^\circ) + \sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta + 120^\circ) \quad [1.114]$$

$$\sin(\theta - 30^\circ) - 2 \sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta + 240^\circ) \quad [1.115]$$

$$\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ) = 0 \quad ; \quad \forall \quad n \in \mathbb{Z} \quad [1.116]$$

Appendix B – Finding Fourier Coefficients of a Phase-Shifted Function

We'll first do the math, and then explain why/when it could be useful. Consider a periodic function with zero mean and period 2π , expressed as a Fourier series:

$$r(\theta) = \sum_{n=1}^{\infty} e_n \sin(n\theta) + f_n \cos(n\theta) \quad [1.117]$$

Let's now introduce a phase shift, ϕ :

$$r(\theta + \phi) = \sum_{n=1}^{\infty} e_n \sin(n\theta + n\phi) + f_n \cos(n\theta + n\phi) \quad [1.118]$$

Using the summation identities in Appendix A, this becomes:

$$r(\theta + \phi) = \sum_{n=1}^{\infty} e_n [\sin(n\theta)\cos(n\phi) + \cos(n\theta)\sin(n\phi)] + f_n [\cos(n\theta)\cos(n\phi) - \sin(n\theta)\sin(n\phi)] \quad [1.119]$$

Now grouping terms:

$$r(\theta + \phi) = \sum_{n=1}^{\infty} \underbrace{[e_n \cos(n\phi) - f_n \sin(n\phi)]}_{e'_n} \sin(n\theta) + \underbrace{[e_n \sin(n\phi) + f_n \cos(n\phi)]}_{f'_n} \cos(n\theta) \quad [1.120]$$

We can see that this is still in the form of a Fourier series, but with the Fourier coefficients modified.

This can come in quite handy in the following scenario. Let's say you are given a winding current waveform as a function of time (either a functional description or actual data points). It is not difficult to derive the Fourier coefficients in either case by either performing the Fourier coefficient integrals in the case of a functional description or performing an FFT in the case of time domain data. Of course, before doing this, you must determine the period of the data. In some cases, this is known before hand by measuring the motor speed. In other cases, the period can be determined by studying the time domain data itself. In either case, coming up with the Fourier coefficients is relatively easy. However, there is the question of how to relate the phase of the data (which is a function of time) to motor position, θ .

Appendix C – Deriving $p(\theta)$ from phase-to-phase open-circuit voltage measurement

Assume what we actually measure is:

$$V_{AB}(\theta) = \varepsilon_A - \varepsilon_B = k_v \dot{\theta} [p(\theta) - p(\theta + 120^\circ)]$$

Expressing these as Fourier series:

$$\begin{aligned} \sum_{n=1}^{\infty} A_n \sin(n\theta) + B_n \cos(n\theta) &= k_v \dot{\theta} \left[\sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) - \sum_{n=1}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + b_n \cos(n\theta + n \cdot 120^\circ) \right] \\ &= k_v \dot{\theta} \sum_{n=1}^{\infty} a_n [\sin(n\theta) - \sin(n\theta + n \cdot 120^\circ)] + b_n [\cos(n\theta) - \cos(n\theta + n \cdot 120^\circ)] \end{aligned}$$

Expanding this using the trig summation identities:

$$\sum_{n=1}^{\infty} A_n \sin(n\theta) + B_n \cos(n\theta) = k_v \dot{\theta} \sum_{n=1}^{\infty} \left\{ \begin{aligned} &a_n \left[\sin(n\theta) - \sin(n\theta) \cos(n \cdot 120^0) - \cos(n\theta) \sin(n \cdot 120^0) \right] + \\ &b_n \left[\cos(n\theta) - \cos(n\theta) \cos(n \cdot 120^0) + \sin(n\theta) \sin(n \cdot 120^0) \right] \end{aligned} \right\}$$

Grouping terms:

$$\sum_{n=1}^{\infty} A_n \sin(n\theta) + B_n \cos(n\theta) = k_v \dot{\theta} \sum_{n=1}^{\infty} \left\{ \begin{aligned} &\left[a_n (1 - \cos(n \cdot 120^0)) + b_n \sin(n \cdot 120^0) \right] \sin(n\theta) + \\ &\left[-a_n \sin(n \cdot 120^0) + b_n (1 - \cos(n \cdot 120^0)) \right] \cos(n\theta) \end{aligned} \right\}$$

Equating summation terms on left and right:

$$A_n = a_n (1 - \cos(n \cdot 120^0)) + b_n \sin(n \cdot 120^0)$$

$$B_n = -a_n \sin(n \cdot 120^0) + b_n (1 - \cos(n \cdot 120^0))$$

Solving this 2x2 system:

$$a_n = \frac{1}{2} A_n - \frac{\sin(n \cdot 120^0)}{2[1 - \cos(n \cdot 120^0)]} B_n \quad [1.121]$$

$$b_n = \frac{1}{2} B_n + \frac{\sin(n \cdot 120^0)}{2[1 - \cos(n \cdot 120^0)]} A_n \quad [1.122]$$

Appendix D - Determining Average Motor Torque from an (almost) Arbitrary Drive Voltage Waveform Including Winding Inductance

Assume the motor controller forces the drive voltages to be of the form:

$$V_{AB} = V_A - V_B = v(\theta) \quad [1.123]$$

$$V_{BC} = V_B - V_C = v(\theta + 120^\circ) \quad [1.124]$$

$$V_{CA} = V_C - V_A = v(\theta + 240^\circ) \quad [1.125]$$

Where:

$$v(\theta) = \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) \quad [1.126]$$

By the way, note that the negative of the summation of [1.36] and [1.37] yields:

$$V_C - V_A = -[v(\theta) + v(\theta + 120^\circ)] \quad [1.127]$$

Comparing this with [1.38], we realize that in order for [1.36] through [1.38] to be satisfied, the following relation must be true:

$$v(\theta + 240^\circ) = -[v(\theta) + v(\theta + 120^\circ)] \quad [1.128]$$

It's not obvious (at least to me) if this automatically satisfied for the most general form of $v(\theta)$. So let's take a look. Note the following:

$$\begin{aligned} v(\theta) + v(\theta + 120^\circ) &= \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) + \sum_{n=1}^{\infty} g_n \sin(n\theta + n \cdot 120^\circ) + h_n \cos(n\theta + n \cdot 120^\circ) \\ &= \sum_{n=1}^{\infty} g_n [\sin(n\theta) + \sin(n\theta + n \cdot 120^\circ)] + h_n [\cos(n\theta) + \cos(n\theta + n \cdot 120^\circ)] \\ &= \sum_{n=1}^{\infty} \left\{ g_n [\sin(n\theta) + \sin(n\theta)\cos(n \cdot 120^\circ) + \cos(n\theta)\sin(n \cdot 120^\circ)] + \right. \\ &\quad \left. h_n [\cos(n\theta) + \cos(n\theta)\cos(n \cdot 120^\circ) - \sin(n\theta)\sin(n \cdot 120^\circ)] \right\} \quad [1.129] \\ &= \sum_{n=1}^{\infty} \left\{ g_n \sin(n\theta) + g_n \cos(n \cdot 120^\circ) \sin(n\theta) + g_n \sin(n \cdot 120^\circ) \cos(n\theta) + \right. \\ &\quad \left. h_n \cos(n\theta) + h_n \cos(n \cdot 120^\circ) \cos(n\theta) - h_n \sin(n \cdot 120^\circ) \sin(n\theta) \right\} \\ &= \sum_{n=1}^{\infty} [g_n + g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ)] \sin(n\theta) + [g_n \sin(n \cdot 120^\circ) + h_n + h_n \cos(n \cdot 120^\circ)] \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \{ g_n [1 + \cos(n \cdot 120^\circ)] - h_n \sin(n \cdot 120^\circ) \} \sin(n\theta) + \{ g_n \sin(n \cdot 120^\circ) + h_n [1 + \cos(n \cdot 120^\circ)] \} \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \{ g_n [1 + \cos(n \cdot 240^\circ)] + h_n \sin(n \cdot 240^\circ) \} \sin(n\theta) + \{ -g_n \sin(n \cdot 240^\circ) + h_n [1 + \cos(n \cdot 240^\circ)] \} \cos(n\theta) \end{aligned}$$

Rearranging, we see that:

$$-[v(\theta) + v(\theta + 120^\circ)] = \sum_{n=1}^{\infty} \{ -g_n [1 + \cos(n \cdot 240^\circ)] - h_n \sin(n \cdot 240^\circ) \} \sin(n\theta) + \{ g_n \sin(n \cdot 240^\circ) - h_n [1 + \cos(n \cdot 240^\circ)] \} \cos(n\theta) \quad [1.130]$$

Also note that:

$$\begin{aligned} v(\theta + 240^\circ) &= \sum_{n=1}^{\infty} g_n \sin(n\theta + n \cdot 240^\circ) + h_n \cos(n\theta + n \cdot 240^\circ) \\ &= \sum_{n=1}^{\infty} [g_n \cos(n \cdot 240^\circ) - h_n \sin(n \cdot 240^\circ)] \sin(n\theta) + [g_n \sin(n \cdot 240^\circ) + h_n \cos(n \cdot 240^\circ)] \cos(n\theta) \end{aligned} \quad [1.131]$$

Seeing that [1.130] and [1.131] are both in the form of a Fourier series, we can compare their Fourier coefficients. And we see by inspection that in order for $-[v(\theta) + v(\theta + 120^\circ)]$ to equal $v(\theta + 240^\circ)$, the following must be satisfied:

$$g_n \left[1 + \cos(n \cdot 240^\circ) \right] = -g_n \cos(n \cdot 240^\circ) \quad [1.132]$$

$$\Rightarrow g_n \left[1 + 2 \cos(n \cdot 240^\circ) \right] = 0$$

and

$$h_n \left[1 + \cos(n \cdot 240^\circ) \right] = -h_n \cos(n \cdot 240^\circ) \quad [1.133]$$

$$\Rightarrow h_n \left[1 + 2 \cos(n \cdot 240^\circ) \right] = 0$$

The quantity $\left[1 + 2 \cos(n \cdot 240^\circ) \right]$ equals zero for $n = 1, 2, 4, 5, 7, 8, \dots$. But it is nonzero (equal to 3) for $n = 3, 6, 9, \dots$. So the only way to satisfy [1.132] and [1.133] for $n = 3, 6, 9, \dots$ is for:

$$g_n = h_n = 0 ; n = 3, 6, 9, \dots \quad [1.134]$$

What this means is that by assuming the drive voltages satisfy [1.36] through [1.38], we are actually imposing constraints on the function $v(\theta)$.

Performing KVL from the A phase to the B phase (including the winding inductance):

$$V_{AB} = L \frac{di_A}{dt} + i_A R + \varepsilon_A - \varepsilon_B - i_B R - L \frac{di_B}{dt} \quad [1.135]$$

And from phase B to C:

$$V_{BC} = L \frac{di_B}{dt} + i_B R + \varepsilon_B - \varepsilon_C - i_C R - L \frac{di_C}{dt} \quad [1.136]$$

Rearranging these:

$$L \frac{d}{dt} (i_A - i_B) + R (i_A - i_B) = V_{AB} - \varepsilon_A + \varepsilon_B \quad [1.137]$$

$$L \frac{d}{dt} (i_B - i_C) + R (i_B - i_C) = V_{BC} - \varepsilon_B + \varepsilon_C \quad [1.138]$$

Introducing two new variables:

$$\alpha \equiv i_A - i_B \quad [1.139]$$

$$\beta \equiv i_B - i_C \quad [1.140]$$

Rewriting [1.137] and [1.138] in terms of these:

$$L \frac{d\alpha}{dt} + R\alpha = V_{AB} - \varepsilon_A + \varepsilon_B \quad [1.141]$$

$$L \frac{d\beta}{dt} + R\beta = V_{BC} - \varepsilon_B + \varepsilon_C \quad [1.142]$$

Note these are two independent first order ODE's. However, the independent variable in them is time, while the forcing functions (on the right) are functions of θ . However, time and position are related (we'll assume by a constant velocity), so we can either express the forcing functions as functions of time, or convert the independent variable to θ . We'll elect to do the latter. Note that:

$$\frac{d\alpha}{dt} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{dt} = \frac{d\alpha}{d\theta} \cdot \dot{\theta} \quad [1.143]$$

Using this to rewrite the ODE's:

$$L\dot{\theta}\frac{d\alpha}{d\theta} + R\alpha = \underbrace{V_{AB} - \varepsilon_A + \varepsilon_B}_{f_\alpha(\theta)} \quad [1.144]$$

$$L\dot{\theta}\frac{d\beta}{d\theta} + R\beta = \underbrace{V_{BC} - \varepsilon_B + \varepsilon_C}_{f_\beta(\theta)} \quad [1.145]$$

Looking first at [1.144], if the forcing is of the form:

$$f_\alpha(\theta) = A\sin(\omega\theta) + B\cos(\omega\theta),$$

Then the steady state solution (after transient response has died out) is:

$$\alpha(\theta) = C(\omega) \cdot \sin(\omega\theta) + D(\omega) \cdot \cos(\omega\theta), \quad [1.146]$$

where:

$$C(\omega) = \frac{AR + BL\dot{\theta}\omega}{R^2 + (L\dot{\theta}\omega)^2}$$

$$D(\omega) = \frac{-AL\dot{\theta}\omega + BR}{R^2 + (L\dot{\theta}\omega)^2}$$

In our case, the forcing function is given in terms of a Fourier series:

$$f_\alpha(\theta) = \sum_{n=1}^{\infty} x_{\alpha,n} \sin(n\theta) + y_{\alpha,n} \cos(n\theta) \quad [1.147]$$

And we're taking advantage of superposition to determine the output. In other words, we use [1.146] to determine the output corresponding to each term in the summation in [1.147]:

$$\alpha_n(\theta) = \frac{x_{\alpha,n}R + y_{\alpha,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \cdot \sin(n\theta) + \frac{-x_{\alpha,n}L\dot{\theta}\omega + y_{\alpha,n}R}{R^2 + (L\dot{\theta}\omega)^2} \cdot \cos(n\theta), \quad [1.148]$$

And the total output is simply the sum of all of these:

$$\begin{aligned} \alpha(\theta) &= \sum_{n=1}^{\infty} \alpha_n(\theta) \\ &= \sum_{n=1}^{\infty} \frac{x_{\alpha,n}R + y_{\alpha,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \sin(n\theta) + \frac{-x_{\alpha,n}L\dot{\theta}\omega + y_{\alpha,n}R}{R^2 + (L\dot{\theta}\omega)^2} \cos(n\theta) \end{aligned} \quad [1.149]$$

We see by inspection that [1.145] is the same ODE as [1.144] with just a different forcing function. So if its input function is given by:

$$f_\beta(\theta) = \sum_{n=1}^{\infty} x_{\beta,n} \sin(n\theta) + y_{\beta,n} \cos(n\theta), \quad [1.150]$$

then its output is given by:

$$\begin{aligned} \beta(\theta) &= \sum_{n=1}^{\infty} \beta_n(\theta) \\ &= \sum_{n=1}^{\infty} \frac{x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \sin(n\theta) + \frac{-x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \cos(n\theta), \end{aligned} \quad [1.151]$$

But what good are α and β to us? What we care about are the three winding currents. We can combine [1.139] and [1.140] with conservation of current to form a system of three equations in three unknowns:

$$\begin{cases} i_A - i_B = \alpha \\ i_B - i_C = \beta \\ i_A + i_B + i_C = 0 \end{cases} \quad [1.152]$$

The solution to this system is:

$$i_A = \frac{1}{3}\beta + \frac{2}{3}\alpha = \frac{\beta + 2\alpha}{3} \quad [1.153]$$

$$i_B = \frac{1}{3}\beta - \frac{1}{3}\alpha = \frac{\beta - \alpha}{3} \quad [1.154]$$

$$i_C = -\frac{2}{3}\beta - \frac{1}{3}\alpha = -\frac{2\beta + \alpha}{3} \quad [1.155]$$

Plugging [1.149] and [1.151] into [1.153], we can express i_A as:

$$i_A = \sum_{n=1}^{\infty} \frac{x_{\alpha,n}R + y_{\alpha,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \sin(n\theta) + \frac{-x_{\alpha,n}L\dot{\theta}\omega + y_{\alpha,n}R}{R^2 + (L\dot{\theta}\omega)^2} \cos(n\theta) + \frac{1}{3} \sum_{n=1}^{\infty} \frac{x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \sin(n\theta) + \frac{-x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \cos(n\theta) \quad [1.156]$$

We can combine the two summations:

$$\begin{aligned} i_A &= \sum_{n=1}^{\infty} \left[\frac{2}{3} \frac{x_{\alpha,n}R + y_{\alpha,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} + \frac{1}{3} \frac{x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \left[\frac{2}{3} \frac{-x_{\alpha,n}L\dot{\theta}\omega + y_{\alpha,n}R}{R^2 + (L\dot{\theta}\omega)^2} + \frac{1}{3} \frac{-x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \frac{1}{3} \left[2 \frac{x_{\alpha,n}R + y_{\alpha,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} + \frac{x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \frac{1}{3} \left[2 \frac{-x_{\alpha,n}L\dot{\theta}\omega + y_{\alpha,n}R}{R^2 + (L\dot{\theta}\omega)^2} + \frac{-x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \\ &= \sum_{n=1}^{\infty} \frac{1}{3} \left[\frac{2x_{\alpha,n}R + 2y_{\alpha,n}L\dot{\theta}n + x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \frac{1}{3} \left[\frac{-2x_{\alpha,n}L\dot{\theta}\omega + 2y_{\alpha,n}R - x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[\frac{2x_{\alpha,n}R + 2y_{\alpha,n}L\dot{\theta}n + x_{\beta,n}R + y_{\beta,n}L\dot{\theta}n}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \left[\frac{-2x_{\alpha,n}L\dot{\theta}\omega + 2y_{\alpha,n}R - x_{\beta,n}L\dot{\theta}\omega + y_{\beta,n}R}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \\ &= \frac{1}{3} \sum_{n=1}^{\infty} \left[\frac{R(2x_{\alpha,n} + x_{\beta,n}) + L\dot{\theta}n(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \left[\frac{-L\dot{\theta}\omega(2x_{\alpha,n} + x_{\beta,n}) + R(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \end{aligned} \quad [1.157]$$

Rewriting for the last time:

$$i_A = \sum_{n=1}^{\infty} \left[\frac{1}{3} \cdot \frac{R(2x_{\alpha,n} + x_{\beta,n}) + L\dot{\theta}n(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \left[\frac{1}{3} \cdot \frac{-L\dot{\theta}\omega(2x_{\alpha,n} + x_{\beta,n}) + R(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta) \quad [1.158]$$

Note that we have expressed i_A in terms of its Fourier series, with the sine and cosine coefficients in square brackets. Summarizing on the following page:

$$\begin{aligned}
V_{AB} &= v(\theta) & V_{BC} &= v(\theta + 120^\circ) & V_{CA} &= v(\theta + 240^\circ) \\
v(\theta) &= \sum_{n=1}^{\infty} g_n \sin(n\theta) + h_n \cos(n\theta) \quad ; \quad (g_n = h_n = 0 \text{ for } n = 3, 6, 9, \dots) \\
\varepsilon_A &= k_v \dot{\theta} \cdot p(\theta) & \varepsilon_B &= k_v \dot{\theta} \cdot p(\theta + 120^\circ) & \varepsilon_C &= k_v \dot{\theta} \cdot p(\theta + 240^\circ) \\
p(\theta) &= \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) \quad ; \quad (a_1 = 1, b_1 = 0) \\
f_\alpha(\theta) &= V_{AB} - \varepsilon_A + \varepsilon_B = \sum_{n=1}^{\infty} x_{\alpha,n} \sin(n\theta) + y_{\alpha,n} \cos(n\theta) \\
f_\beta(\theta) &= V_{BC} - \varepsilon_B + \varepsilon_C = \sum_{n=1}^{\infty} x_{\beta,n} \sin(n\theta) + y_{\beta,n} \cos(n\theta) \\
i_A &= \sum_{n=1}^{\infty} \left[\frac{1}{3} \cdot \frac{R(2x_{\alpha,n} + x_{\beta,n}) + L\dot{\theta}n(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}n)^2} \right] \sin(n\theta) + \left[\frac{1}{3} \cdot \frac{-L\dot{\theta}\omega(2x_{\alpha,n} + x_{\beta,n}) + R(2y_{\alpha,n} + y_{\beta,n})}{R^2 + (L\dot{\theta}\omega)^2} \right] \cos(n\theta)
\end{aligned}$$

As a sanity check, note what happens as the ratio (L/R) approaches zero (i.e. the winding inductance becomes negligible). In this scenario, i_A simplifies to:

$$i_A = \frac{1}{3R} \sum_{n=1}^{\infty} (2x_{\alpha,n} + x_{\beta,n}) \sin(n\theta) + (2y_{\alpha,n} + y_{\beta,n}) \cos(n\theta)$$

Now expanding $x_{\alpha,n}$ and $x_{\beta,n}$:

$$\begin{aligned}
x_{\alpha,n} &= g_n - k_v \dot{\theta} a_n + k_v \dot{\theta} [a_n \cos(n \cdot 120^\circ) - b_n \sin(n \cdot 120^\circ)] \\
&= g_n + a_n k_v \dot{\theta} [-1 + \cos(n \cdot 120^\circ)] - b_n k_v \dot{\theta} \sin(n \cdot 120^\circ) \\
x_{\beta,n} &= [g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ)] - k_v \dot{\theta} [a_n \cos(n \cdot 120^\circ) - b_n \sin(n \cdot 120^\circ)] + k_v \dot{\theta} [a_n \cos(n \cdot 240^\circ) - b_n \sin(n \cdot 240^\circ)] \\
&= g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ) + a_n k_v \dot{\theta} [-\cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] + b_n k_v \dot{\theta} [\sin(n \cdot 120^\circ) - \sin(n \cdot 240^\circ)]
\end{aligned}$$

Combining:

$$\begin{aligned}
2x_{\alpha,n} + x_{\beta,n} &= 2g_n + a_n k_v \dot{\theta} [-2 + 2\cos(n \cdot 120^\circ)] - b_n 2k_v \dot{\theta} \sin(n \cdot 120^\circ) + g_n \cos(n \cdot 120^\circ) - h_n \sin(n \cdot 120^\circ) + a_n k_v \dot{\theta} [-\cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] + b_n k_v \dot{\theta} [\sin(n \cdot 120^\circ) - \sin(n \cdot 240^\circ)] \\
&= [2 + \cos(n \cdot 120^\circ)] g_n - h_n \sin(n \cdot 120^\circ) + a_n k_v \dot{\theta} [-2 + 2\cos(n \cdot 120^\circ) - \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] + b_n k_v \dot{\theta} [\sin(n \cdot 120^\circ) - \sin(n \cdot 240^\circ) - 2\sin(n \cdot 120^\circ)] \\
&= [2 + \cos(n \cdot 120^\circ)] g_n - h_n \sin(n \cdot 120^\circ) + a_n k_v \dot{\theta} [-2 + \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ)] - b_n k_v \dot{\theta} [\sin(n \cdot 120^\circ) + \sin(n \cdot 240^\circ)]
\end{aligned}$$